IMSC 2058 Solution for Homework 6

Ex 4.24

We first show that if each X_i is Hausdorff, then X is Hausdorff.

Let $p = (p_i)_{i\in I}$ and $q = (q_i)_{i\in I}$ be distinct points in X, so there exists at least one index $j \in I$ with $p_j \neq q_j$. Since X_j is Hausdorff, there exist two disjoint open sets $p_j \in U_j$ and $q_j \in V_j$ in X_j with $U_j \cap V_j = \emptyset$.

Define $U = \prod_{i\in I} U_i$ and $V = \prod_{i\in I} V_i$, where $U_i = V_i = X_i$ for $i \neq j$. Clearly, U, V are open and $p \in U$, $q \in V$, moreover $U \bigcap V = \emptyset$. Hence $X = \prod_{i\in I} X_i$ is Hausdorff.

If $X = \prod_{i\in I} X_i$ is Hausdorff. Suppose, for contradiction, that some X_j (for fixed $j \in I$) is not Hausdorff.

Let $c = (c_i)_{i\in I} \in X$ be an arbitrary point. Define two points $p, q \in X$ by

$$p = (c_1, c_2, \cdots, a, c_{j+1}, \cdots), \quad p = (c_1, c_2, \cdots, b, c_{j+1}, \cdots)$$

where $a \neq b$. Clearly, p, q agree in every coordinate except j. Since X is Hausdorff, there exist disjoint open neighborhoods U, V such that $p \in U$, $q \in V$ and $U \bigcap V = \emptyset$.

We write $U = \prod_{i\in I} U_i$ and $V = \prod_{i\in I} V_i$, where each U_i, V_i is open in X_i, and all but finitely many are full spaces.

Since the projections $\pi_i : X \to X_i$ are continuous. Then $\pi_j(U)$ and $\pi_j(V)$ are open in X_j and $a \in \pi_j(U)$, $b \in \pi_j(V)$.

If $\pi_j(U) \cap \pi_j(V) = \emptyset$, then U_j and V_j would separate a and b in X_j, contradicting the assumption that X_j is not Hausdorff. Thus $\pi_j(U) \cap \pi_j(V) \neq \emptyset$. There must exist $d \in \pi_j(U) \cap \pi_j(V)$.

Let $r = (c_1, c_2, \cdots, d, c_{j+1}, \cdots)$. Then $r \in U$ and $r \in V$,which means that $U \cap V \neq \emptyset$. It contradict to $U \cap V = \emptyset$, so every X_i must be Hausdorff.

Ex 5.4

For each $x \in X$, by continuity of f at x, there exists $\delta_x > 0$ such that if $\rho(x, y) < \delta_x$, then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. We define the open ball $B(x, \delta_x) = \{y \in X \mid \rho(x, y) < \delta_x\}$, which is open in the metric topology. Then the collection $\{B(x, \delta_x) \mid x \in X\}$ is an open cover of X. Since X is compact, this open cover has a finite subcover

$$X = \bigcup_{k=1}^{n} B(x_k, \delta_{x_k})$$

where $x_1, x_2, \ldots, x_n \in X$.

Let $\delta = \min\{\delta_{x_1}/2, \delta_{x_2}/2, \ldots, \delta_{x_n}/2\} > 0$. Now, for arbitrary $x, y \in X$ with $\rho(x, y) < \delta$, y must lie in at least one $B(x_k, \delta_{x_k}/2)$ for some k. Then

$$\rho(x, x_k) \leq \rho(x, y) + \rho(y, x_k) < \delta + \frac{\delta_{x_k}}{2} \leq \delta_{x_k}.$$

It follows that

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore f is uniform continuous on X.